*On Certain Orthogonal Systems of Lines and the Problem of Determining Surfaces Referred to Them.**

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§ 1. *Introduction.*

Bonnet has shown that the problem of determining surfaces referred to orthogonal lines can be reduced to the solution of the following systems of equations: †

$$\left.\begin{aligned} &\frac{\partial M}{\partial v}-\frac{\partial N}{\partial u}-PQ-RS=0, \\ &\frac{\partial P}{\partial v}-\frac{\partial S}{\partial u}+MQ+NR=0, \\ &\frac{\partial R}{\partial v}+\frac{\partial Q}{\partial u}+MS-NP=0, \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} &\frac{R}{\sqrt{E}}+\frac{S}{\sqrt{G}}=0, \\ &M=-\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}, \\ &N=\quad\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}, \end{aligned}\right\} \quad (2)$$

where $R^2+P^2=e$, $PS-QR=f$, $S^2+Q^2=g$, and where the first and third quadratic forms for the corresponding surface are respectively:

$$\text{I.} \quad \overline{ds}^2=Edu^2+Gdv^2,$$

$$\text{III.} \quad \overline{d\sigma}^2=edu^2+2fdudv+gdv^2.$$

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† *Journal de l'École Polytechnique*, Vol. XLII, pp. 132–151.

To every set of values of the functions P, Q, R, S, M, N, E, and G satisfying (1) and (2) there corresponds a surface S, uniquely determined as Bonnet has shown,* whose cartesian coordinates can be obtained in the usual way as functions of u and v.

In the application of the general method, Bonnet, in every instance, limited himself to the case where the lines of reference were lines of curvature on the surface, and even then he considered the problem only in very particular cases.*

The general problem of determining surfaces referred to lines of curvature has been treated by us in a previous paper † from the standpoint of the Bonnet equations.

The present paper has to do with the problem of determining surfaces referred to *any set of orthogonal lines.* Assuming the general results obtained by Bonnet, we have sought to determine those systems of orthogonal lines whose choice as lines of reference would either simplify the general problem of determining surfaces, or else enable us to obtain by algebraic operations from every surface found, another, which is, it appears, an associate of the former.

In § 2 the problem of solving the systems of equations (1) and (2) is shown to hinge on the solution of a Laplace equation, if we consider M and N as known, whose invariants involve these latter functions and their derivatives. This result suggests the choice of such lines as lines of reference as will cause an invariant of this Laplace equation, or of one obtained from it by the method of Laplace, to vanish. For then and only then, in general, can the Laplace equation be solved. The lines thus defined we have called *invariant lines.* There are n sets of such lines on every surface where n is arbitrarily large. If, in particular, the invariant lines are also lines of curvature, the surfaces corresponding compose the general class obtained by the solution of the problem ‡ of the spherical representation, the problem discussed in our previous paper. §

In § 3 we have determined the equation which defines those orthogonal systems of lines on a surface S which correspond to orthogonal systems on the associates of S. Such lines we have called *principal orthogonal lines*, or more briefly, *principal lines.* Their choice as lines of reference makes possible the determination by algebraic operations of the associates of S, and hence its infinitesimal deformations.

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* See reference above.

† See reference below.

‡ See Darboux, *Théorie des Surfaces*, Vol. IV, p. 169.

§ See reference below.

In § 4 we have obtained a new method for determining whether a given surface is isothermic, which is applicable whenever the lines of reference are orthogonal. We have also extended the Bour-Darboux theorem for finding a sequence of isothermic surfaces from a given isothermic surface so that, while before it was applicable only when the lines of reference on the known surface were lines of curvature, it is now applicable when the lines of reference are any set of orthogonal lines.

2. *Invariant Lines.*

Eliminating E and G from (2) we have

$$\frac{\partial^2 \log\left(\frac{S}{R}\right)}{\partial u \partial v} + \frac{\partial}{\partial u}\left(M\frac{S}{R}\right) + \frac{\partial}{\partial v}\left(N\frac{R}{S}\right) = 0, \tag{3}$$

which, with (1), gives four equations for the determination of the functions M, N, P, Q, R, and S.

The problem of determining surfaces referred to orthogonal lines reduces, therefore, to the simultaneous solution of (1) and (3), the functions E and G of the first quadratic form being obtained then by quadrature from the following:

$$\left.\begin{aligned} \frac{\partial \log \sqrt{E}}{\partial u} &= -\frac{\partial}{\partial u}\log\left(\frac{S}{R}\right) - N\left(\frac{R}{S}\right), \\ \frac{\partial \log(\sqrt{E})}{\partial v} &= \quad M\left(\frac{S}{R}\right), \\ \sqrt{G} &= -\frac{\frac{\partial\sqrt{E}}{\partial v}}{M}. \end{aligned}\right\} \tag{4}$$

In (1) and (3) we have six functions involved in four equations. If we add to this system a fifth equation involving one or more of the functions and their derivatives, we shall not, in general, limit the result so far as surfaces are concerned,* but shall limit the choice of orthogonal lines as lines of reference on them.

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* An exception to the rule would arise if we should use the equation $\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}$.

For example, if we associate with (1) and (3) the equation

$$(1^\circ)\quad M=0,$$

we limit the choice of lines of reference to geodesics and geodesic parallels. We say that this equation *defines* such lines. Again, if we associate with these equations the following,

$$(2^\circ)\quad \frac{\partial M}{\partial u}+\frac{\partial N}{\partial v}=0,$$

we limit our choice of lines to a class which includes isothermal lines as a particular case.

In the preceding examples we have used an equation involving the functions M and N in defining lines of reference. If in either case we had used also another equation involving the same functions, we might have said that the last equation added, defined a certain class of surfaces referred to the lines defined by the first. For example, if we had used with (1) and (3) the following,

$$\text{(i)}\quad M=0,$$
$$\text{(ii)}\quad N=f(u),$$

we might have said that (i) defined as lines of reference geodesics and geodesic parallels, and that (ii) defined a certain class of surfaces referred to such lines, namely surfaces of revolution and surfaces applicable to them.

Moreover, it appears from (2) that the determination of the functions M and N by two equations, as in the preceding examples, amounts to the determination of E and G, and hence of the first fundamental form for the surfaces sought.

We proceed to point out certain systems of lines whose choice as lines of reference simplifies the problem of solving (1) and (3), if M and N are assumed to be known, and hence that of determining the corresponding surfaces.

The equation offering the greatest difficulties of solution seemingly is (3). But this can be replaced by

$$\frac{\partial^2\phi}{\partial u\partial v}-\frac{\partial \log M}{\partial u}\frac{\partial\phi}{\partial v}+MN\phi=0, \tag{5}$$

making use of the substitution

$$\frac{S}{R}=\frac{1}{\phi}\left(-\int N\phi du+V\right),$$

where V is an arbitrary function of integration.

This is a Laplace equation, and the conditions under which its general solution can be obtained are known.*

Starting with (5) and forming the sequence of Laplace equations in the one way, we have as the invariants of this equation, and the principal invariants of the others written in order, the following:

$$\left.\begin{aligned}
k &= -\frac{\partial^2 \log M}{\partial u \partial v} - MN, \quad h = -MN, \\
k_1 &= -\frac{\partial^2 \log (Mk)}{\partial u \partial v} - k, \\
k_2 &= -\frac{\partial^2 \log (Mkk_1)}{\partial u \partial v} - k_1, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
k_{i+1} &= -\frac{\partial^2 \log (Mk\ldots.k_i)}{\partial u \partial v} - k_i, \quad \text{where } i = 0, 1, 2, \ldots.
\end{aligned}\right\}$$

The general solution of (5) can be obtained as a function of M and N, and arbitrary functions of integration when and only when, in general, some one of these invariants vanish.

We take, therefore, as the defining equations† for our lines of reference,

$$\left.\begin{aligned}
&h = 0, \text{ or} \\
&k_i = 0, \text{ where } i \text{ has the values } 0, 1, 2, 3, \ldots.
\end{aligned}\right\} \quad (6)$$

The problem of solving (1) and (3) is now reduced to the solution of three equations in P, Q, and R, two of which are linear in these, expressing S in terms of known functions and R, the other being an algebraic equation.

The lines defined by (6) we shall call *invariant lines*, as they arise from equating the invariants of Laplace equations to zero.

These lines appear upon all surfaces and are related in this way.‡ The lines defined by $M = 0$ invert into lines included in the class defined by $k = 0$; and, in general, the lines defined by $k_i = 0$ are either lines obtained by inversion from lines defined by $k_{i-1} = 0$, or $k_{i+1} = 0$.

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* Darboux, *Théorie des Surfaces*, Vol. II, pp. 22–30.

† See previous reference, p. 30.

‡ See work of paper referred to just below. The relation between the systems of lines is the same, whether they are lines of curvature or not.

If, in order to particularize the surfaces referred to any set of invariant lines, we use as the added condition, that the lines of reference (invariant lines) shall be lines of curvature on the surface, we obtain some very interesting surfaces. They are in fact the very surfaces which arise from the solution of the problem of the spherical representation proper. The general class, as we have shown,* can be obtained by quadrature and algebraic operations from a particular set having one set of their lines of curvature plane, geodesic lines.

§ 3. *Principal Lines.*

If the functions M, N, P, Q, R, and S satisfy (1) and (3), then likewise the functions M_0, N_0, etc., defined as follows:

$$\left.\begin{aligned} M_0 &= -M, \\ N_0 &= -N, \\ P_0 &= R, \\ Q_0 &= -S, \\ R_0 &= P, \\ S_0 &= -Q, \end{aligned}\right\} \tag{7}$$

will also satisfy (1). For evidently the first equation is satisfied and the second and third interchange when these values are substituted. If, therefore, the ratio of S_0 over R_0 satisfies (3) when M and N are replaced by their negatives, or, in other words, if the ratio of Q over P satisfies (3), M and N retaining their original values, then, to the surface S corresponding to the first solution, there will be another S_0 corresponding to the second, and this last surface will have the same spherical representation of its lines of reference as the first, since they will both have the same third form. Moreover, it is easily shown that the asymptotic lines on one surface correspond to a conjugate system on the other, and vice versa. Hence S and S_0 are *associate surfaces*, and the orthogonal lines of reference on S correspond to orthogonal lines on S_0.

Such lines we have called *principal orthogonal lines*, or, for brevity, *principal lines*.

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* A. E. Young, "On the Problem of the Spherical Representation, etc.," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII, pp. 37–51.

In view of the preceding work we may take as the defining equation of *principal* lines

$$\frac{\partial^2 \log\left(\frac{Q}{P}\right)}{\partial u \partial v} + \frac{\partial}{\partial u}\left(M\frac{Q}{P}\right) + \frac{\partial}{\partial v}\left(N\frac{P}{Q}\right) = 0,$$

which, of course, is nothing but (3) with S over R replaced by Q over P.

There are an infinity of principal lines on every surface, corresponding respectively to the infinity of associates which the surface has. For every set of associate surfaces has one (and in general only one) set of orthogonal lines corresponding, and these, as we shall show in the next article, are always defined by an equation similar to (8), when they are chosen for lines of reference, unless the surfaces are isothermic, and related according to the Bour-Darboux theorem.

As soon as a set of *principal* lines are chosen for lines of reference on a given surface, an associate of it can be found by quadrature, and hence:*

The problem of determining the associates of a surface S is the same as that of determining all the systems of principal lines on S.

In closing this article, we may remark that should we use (8) with (1) and (3) in determining surfaces, we would obtain them in pairs, one being the associate of the other, and the lines of reference on both being *principal* lines.

§ 4. *Isothermic Surfaces and Their Determination.*

It remains for us to determine whether there are ways, other than the one noted in the previous article, in which two associate surfaces referred to orthogonal lines can be related, that is in regard to their fundamental functions.

Suppose that S_0 is an associate of S referred to orthogonal lines, and that its fundamental functions are M, N, P, etc.

If it is to have the same third form† as S, then

$$P_0^2 + R_0^2 = P^2 + R^2, \quad P_0 S_0 - Q_0 R_0 = PS - QR, \quad Q_0^2 + S_0^2 = Q^2 + S^2, \quad (9)$$

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* Or, since the determination of an associate carries with it the infinitesimal deformation of the original surface, we have the following: The infinitesimal deformation of a surface S is the same problem as the determination of the *principal* systems on S. See the Theorem of Cosserat, Eisenhart, *Differential Geometry*, p. 380.

† See relations given just below system of equations (2).

and if the asymptotic lines on one are to correspond to a conjugate system on the other,* then

$$PRQ_0 S_0 + 2RSR_0 S_0 + QSP_0 R_0 = 0. \tag{10}$$

Considering the first set of functions as given, we have in (9) and (10) four algebraic equations for the determination of four functions, P_0, Q_0, R_0 and S_0. The work of this simple algebraic problem, which is quite long, will be omitted. Eliminating† three of the functions we have a quadratic equation in P_0^2. Solving this and writing down the corresponding possible values for the other functions, we have, if we omit the solution of the preceding article,

$$\left.\begin{aligned} P_0^2 &= \frac{[P(QR+PS)+2SR^2]^2}{(QR+PS)^2+4R^2S^2}, \\ Q_0^2 &= \frac{[Q(QR+PS)+2RS^2]^2}{(QR+PS)^2+4R^2S^2}, \\ R_0^2 &= \frac{R^2(QR-PS)^2}{(QR+PS)^2+4R^2S^2}, \\ S_0^2 &= \frac{S^2(QR-PS)^2}{(QR+PS)^2+4R^2S^2}. \end{aligned}\right\} \tag{11}$$

If we take the ratio of S_0 over R_0 we find that it is the same as S over R, and it follows from the relation of these ratios to those of E_0 over G_0 and E over G, that every set of orthogonal lines on the original surface would correspond to a similar set on any other surface whose functions are related to the former in this way.

It easily follows that the lines of curvature on one correspond to similar lines on the other, and, as one maps conformally on the other, the surfaces are isothermic surfaces related according to the Bour-Darboux theorem.‡

We conclude, therefore, that *if two associate surfaces are referred to orthogonal lines which correspond, their fundamental functions (the parameters being properly chosen) are related as in* (7) *unless they are both isothermic surfaces related according to the Bour-Darboux theorem, in which case their functions are related as in* (11), *provided the lines of reference are not lines of curvature.*

* This is Bianchi's equation $D''D_0 + DD_0'' + 2D'D_0' = 0$, in our notation. (Bianchi, Lukat's edition, p. 293.)

† We exclude the case where the surfaces are developable, and likewise the case where the lines of reference are lines of curvature.

‡ Darboux, *loc. cit.*, Vol. II, p. 243.

This result gives us a new method of determining whether a surface is isothermic without any consideration of the form which the linear element takes when the lines of reference are lines of curvature. Its advantage lies in the fact that it is applicable whenever the lines of reference are *any set of orthogonal lines*.

Suppose we are given any surface S referred to orthogonal lines and wish to determine whether it is isothermic or not. We have merely to form the functions P_0, Q_0, R_0, and S_0 in terms of the corresponding functions of the given surface according to (11). We can then determine the corresponding values of M_0 and N_0 algebraically from the last two equations of (1), where of course P, etc., have been replaced by P_0, etc. If the new set of values of the six functions, M_0, N_0, etc., satisfy (3) and the first equation of (1), the given surface has a conformal associate and hence is an isothermic surface.*

There are few classes of surfaces which have received more attention from geometers, in respect to their determination, than those which have isothermal lines of curvature (or isothermic surfaces, as they have been named by Darboux.†) The theorem due to Bour and extended in application by Darboux, whereby from a given isothermic surface referred to lines of curvature an infinite sequence of such surfaces can be obtained by algebraic operations and quadrature, has yielded the most valuable results. The limitation of the theorem, that it was applicable only when the lines of reference were lines of curvature, was not in evidence before, as no test‡ existed for determining whether a given surface was isothermic except reference to lines of curvature.

In view of this new test, which is applicable whenever the lines of reference are any orthogonal lines, it would be advantageous to remove the limitation. For although the equation defining the lines of curvature is generally known, in general it is impossible to find the integrating factor which would enable one to change to them as lines of reference. This can be done, and the application of the theorem, extended, may be described as follows:

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*On an isothermic surface any orthogonal system of lines corresponds to orthogonal lines on an associate, but certain systems correspond to orthogonal lines on two associates (one of which is an isothermic surface of course), and these are the *principal orthogonal lines* in this case.

† Darboux, *loc. cit.*, Vol. IV, p. 217.

‡ That is, no test which could be applied to the fundamental functions for a surface in their usual forms.

Being given the cartesian coordinates and fundamental functions M, N, etc., for a surface S referred to any set of orthogonal lines, we may apply our test, and if it is found to be isothermic, write out the corresponding functions for its isothermic associate, which we will refer to as S_0. Obtaining the cartesian coordinates for this last surface in the usual way by quadrature, and applying the principle of inversion to it and the original surface, we have two new ones, S'_0 and S' respectively. As before, by algebraic operations and quadrature we can obtain the fundamental functions and cartesian coordinates of the isothermic associates of these last two surfaces. Inversion applied to the surfaces last obtained gives two other isothermic surfaces, etc., etc.

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